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Dissertation

A MODIFICATION OF MINKOWSKI'S FOUR-DIMENSIONAL SPACE-TIME CONTINUUM CONSISTENT WITH THE RESULTS OF DAYTON C. MILLER'S REPETITION OF THE MICHELSON-MORLEY EXPERIMENT

by

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CROSS REFERENCE SYSTEM

Cross references in this dissertation are in terms of page numbers and fractions thereof. For example, 27.6 means six-tenths of the distance down on page twenty-seven. Equations will likewise be located in this way.

For convenience, some of the symbols employed are herewith listed. Symbols which are utilized in an argument and discarded at its completion are not included.

c = Velocity of light in free space as determined by an observer at rest.
x, y, z, coordinates.
n = Index of refraction.
λ = Wave-length.
m = Mass.
k = 0.05.
$k' = 1 - k$
v, v', v'', velocities of observers relative to space.
$r^2 = x^2 + y^2 + z^2$
t, t', t'', local times.
$i^2 = -1$
$w = ict$
$ds^2 = dw^2 + dr^2$
$d\tau = ds/ic$
$\lambda^2 = c^2 - k^2v^2$
$\mu^2 = c^2 - kv^2$
$\nu^2 = c^2 - v^2$
$\nu_{12}^2 = c^2 - v_1v_2$
$\lambda_{12}^2 = c^2 - k^2v_1v_2$
$\mu_{12}^2 = c^2 + kv_1v_2$
$\rho_{12}^2 = \mu_1^2 - k'v_1v_2$
$v_{12} = v_1 - v_2$

INTRODUCTION

When one speaks of the Michelson-Morley experiment, he generally has reference to the celebrated attempt to detect the motion of the earth through space, and while this is the experiment to which the title refers, yet it is interesting first to consider another experiment performed by the same two men. This was a check up on an experiment performed in 1859 by Fizeau, who in turn was attempting to check a formula proposed by Fresnel.

Fresnel argued that if the velocity of light be measured in a medium the index of refraction of which is n and the velocity of which relative to the observer is v, the velocity of light relative to the observer should be $c/n \pm (1 - 1/n^2)v$, where c as usual represents the velocity of light in free space. In order to decide between Fresnel's formula and the more obvious one, $c/n \pm v$, Fizeau split and recombined a beam of light so that the two halves of the beam passed through two streams of water flowing in opposite directions. By observing the shift of the interference pattern as the speed of the water was varied, Fizeau was able to show that Fresnel's formula was the better of the two.

Michelson and Morley, in 1886, repeated Fizeau's experiment with much better apparatus and completely verified Fresnel's formula.

It is not necessary to discuss Fresnel's derivation of this formula, because it involves unnecessary hypotheses, and also because it follows anyhow from the modern theory of relativity that light will obey Fresnel's formula in passing through a moving medium.

There is, however, more to be said about the history of this formula. In 1895, Lorentz showed that the parenthesis should contain an extra term

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and read, $(1 - 1/n^2 - (l/n)(dn/dl))$, where l is the wave-length of the light involved. This correction takes account of the Doppler effect. Although Michelson and Morley had already verified the formula without the extra term, nevertheless, in 1915, Zeeman again repeated the experiment and this time verified Lorentz' modification. The writer will show presently that still another modification should be made to this long-suffering formula, so that it is to be hoped that still another experimenter will arise to perform the experiment yet more accurately!

The famous experiment of Michelson and Morley is too well known to require much description. A beam of light is split, part being made to traverse a path back and forth at right angles to the supposed motion of the apparatus through space, and the other part a path in line with this motion. The two beams having traveled an equal optical distance are then recombined. If there is any effect of the motion on the two paths, it will be such as to increase both paths, the greater increase being on the path parallel to the motion, and consequently to produce a shift in a system of interference fringes.

In speaking of this experiment, the statement is usually made that absolutely no motion relative to space was detected. This is not true, however, for an effect, about one-twentieth of what was expected, was noted. But the fraction 1/20 is so much nearer to zero than it is to unity, that it was attributed to errors and the experiment was regarded as having a null result. Morley, however, was not satisfied, and continued the experiment later with D. C. Miller as collaborator. Upon the death of Morley in 1923, Miller continued the work alone, reporting upon its progress from time to time. Finally, in July, 1933, in an article covering thirty-

nine pages of the Reviews of Modern Physics, Miller gave a summary of an enormous number of observations and stated his conclusions.

Miller concludes that not only the solar system, but our galactic system as well, is moving "south-ward" through space. The speed of the solar system is 208 kilometers per second toward a point the right ascension and declination of which turn out to be 4h 56m and -70° 30'. This does not contradict the well known fact that the solar system has a motion relative to our galaxy nearly in the opposite direction, toward the constellation Hercules.

In this statement, the orbital motion of the earth has been averaged out. However, Miller's observations prove that even the orbital motion of the earth can be detected. The unexpected feature of the situation is that the experiments give the orbital velocity of the earth correct in direction but not in numerical value. The latter needs to be multiplied by a factor of twenty in order to check with the astronomical value. Indeed it is the known value of the earth's orbital velocity which furnishes the calibration of Miller's method and enables him to arrive at the resultant speed mentioned above.

The writer will assume that Miller's experimental results are approximately correct and will hereafter use the term "absolute velocity" in the sense of velocity relative to space as a quantity capable of experimental determination by Miller's experiment.

Einstein's work on relativity was based on the supposed null result of the Michelson-Morley experiment, but has since found so much other experimental verification that its soundness may fairly be said to depend

more upon the variation of the mass of a high speed electron with its velocity, or the bending of light rays as they pass through the sun's gravitational field, or the displacement of spectral lines toward the red in a strong gravitational field, to say nothing of the fact that his theory also gave the first good explanation of the motion of the perihelion of Mercury, and furnished a derivation for Fresnel's formula given above.

If we let a factor of about 0.05 be represented by the letter k, we can make three contradictory statements about the apparent velocity of light as measured on a system moving in the direction of the propagation of the light with an absolute velocity v, letting c represent the absolute velocity of light as defined above. According to Newtonian "relativity" the velocity of light would be c - v. According to Einstein it would be simply c. Miller now shows that it is c - kv. Since k is so much nearer to zero than it is to unity, any experimental test designed to distinguish between the Galilei (or Newtonian) transformation and the Lorentz (or Einstein) transformation would of course favor the latter.

Transformation equations, however, are not the sole property of the theory of relativity. It is possible to derive a set of transformation equations involving the parameter k which will reduce to those of Lorentz when k equals zero, or to those of Galilei when k is unity. The new equations therefore represent a sort of quasi-relativity. The writer has undertaken the task of deriving these equations and investigating their consequences.

THE CONTRIBUTION OF MINKOWSKI

Minkowski's contribution to the theory of relativity was to give it a geometrical interpretation which makes it rather easier to grasp.

If we suppose space to consist of three dimensions of which we are conscious together with a fourth dimension along which our consciousness is traveling with the speed of light, then, for a complete description of events, we need, in addition to the directions x, y, and z, a direction w which is proportional to ct, where c is the velocity of light and t is time. w is not taken as equal to ct, but merely proportional to it. The proportionality factor is, curiously enough, i, the square root of minus one. There is no particular necessity for visualizing i; it is sufficient to regard it as a numerical factor, inserted to make the equations more symmetrical.

Armed with this four-dimensional machinery we can now put ourselves into the position of a five-dimensional observer whom we shall call A, and stand off and watch the progress of B, the three-dimensional consciousness, as it moves along its four-dimensional track. B will perceive at each instant a single cross-section (three-dimensional) of the whole, and what appears to it as motion is simply the presentation at each instant of a new cross-section.

We shall in what follows let r stand for $(x^2 + y^2 + z^2)^{0.5}$, or, what amounts to the same thing, let y = z = 0, and identify r with x. This will allow us to visualize r plotted against w (which is ict) as a two-dimensional concept. We can rotate the three space axes without disturbing the time axis; during the rotation r will remain constant in both direction and mag-

nitude.

The inevitable motion of B along the time axis alone may be represented by $r = 0$, $w = ict$. If B in addition to this imaginary motion should take on a uniform motion in the direction of r (which is perpendicular to w) with a component of velocity equal to v, in accordance with the equation $r = vt$, then the total four-dimensional displacement would be s, such that $s^2 = r^2 + w^2$.

The conception is simple enough to grasp until the numerical values are inserted, and even then the picture should be retained for its geometrical utility. The awkward fact in connection with the numerical part of the illustration is that s^2 must be considered positive when v is greater than c, a physical impossibility, zero when $v = c$, and negative when v is less than c. The alternative is to deal with hyperbolic space; indeed it would be fair to say that that is what we have anyway in everything but the form of the equations.

These values of s may be easily checked. Let $v = c$; then $r = ct$, $r^2 = c^2t^2$, $r^2 + i^2c^2t^2 = 0$, or $r^2 + w^2 = 0 = s^2$. If $v^2 = c^2 \pm q^2$ where q is real, then $r^2 = v^2t^2 = c^2t^2 \pm q^2t^2$, $r^2 + i^2c^2t^2 = \pm q^2t^2$, $r^2 + w^2 = s^2 = \pm q^2t^2$. That is, s^2 is positive when v^2 is greater than c^2 and negative when v^2 is less than c^2.

The equation $r^2 + w^2 = s^2$ may be visualized as a hyper-sphere the radius of which is imaginary when $v < c$ and real when $v > c$. $r^2 + w^2 = 0$ may be thought of as a hyper-cone the opposite elements of which are normal to each other. The vertex of this cone is the center of the spheres just mentioned. The portion of the sphere within the cone is the part

having imaginary radii; that outside (into which B can never go) has real radii. See figure 1.

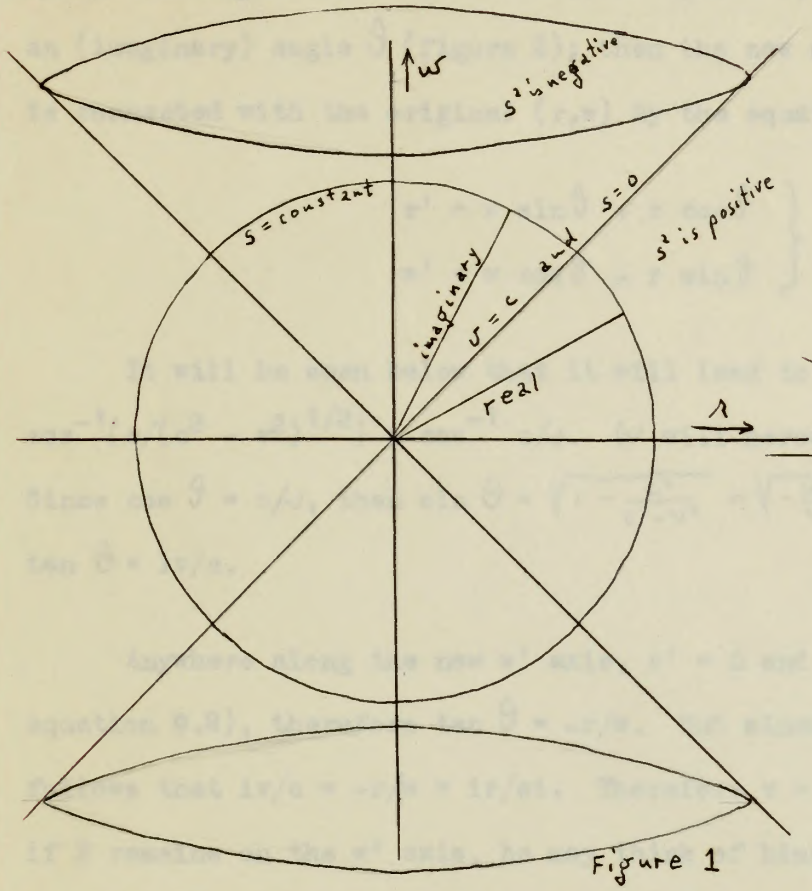


Figure 1

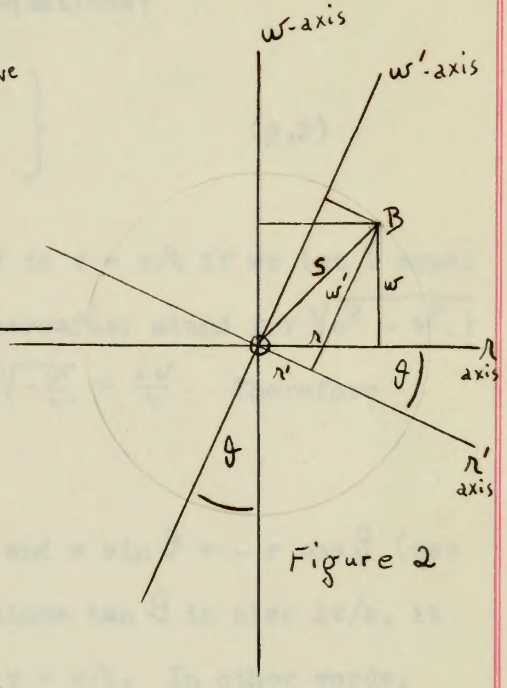
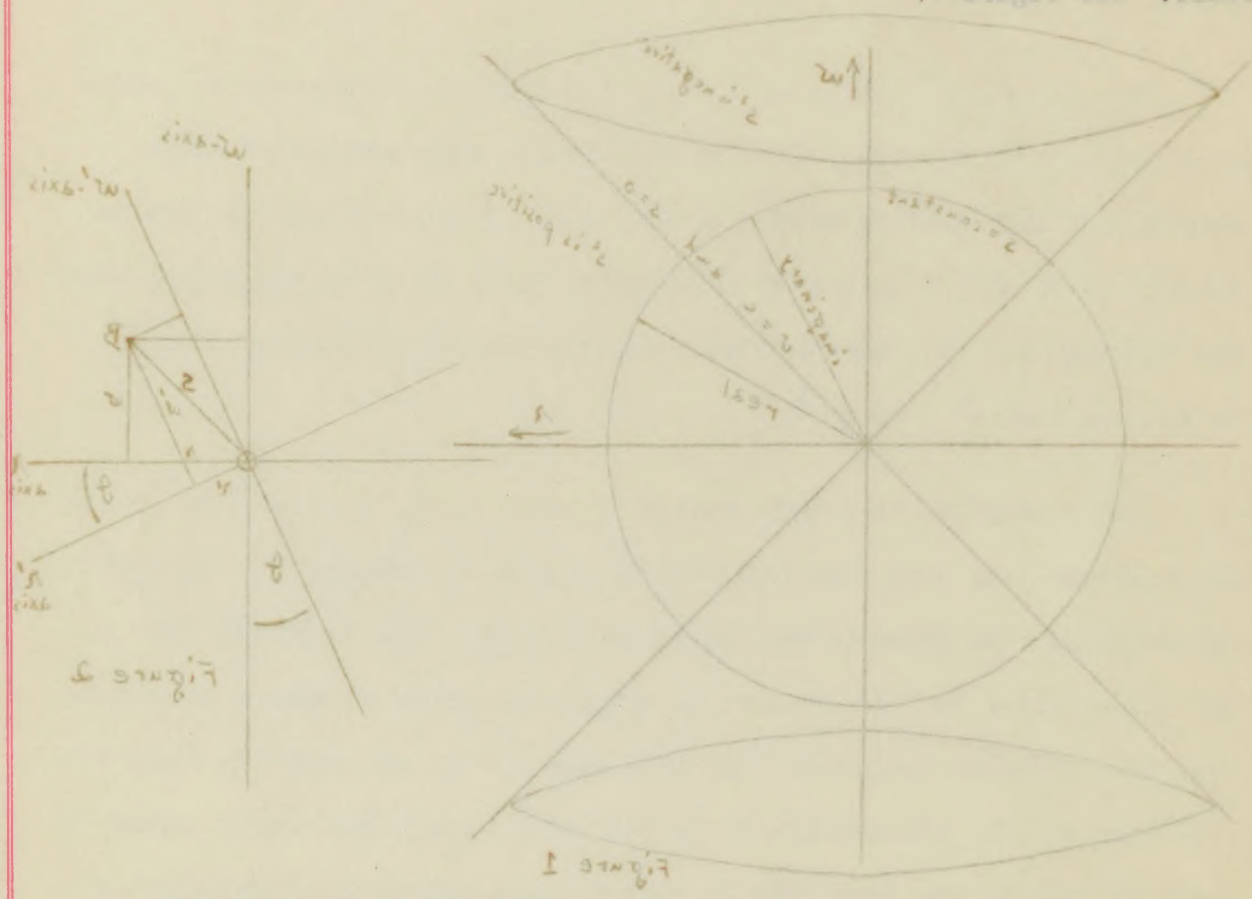


Figure 2

The four-dimensional distance s is called the proper time function and τ (= s/ic) is usually called the proper time. s is the distance, as seen by A (page 6.6) between B and the origin. See figure 2. This whole discussion is of course based on the supposition that both time and distance are reckoned from the origin.

Now let B have a uniform speed, v, that is physically possible (not more than c) and assume the origin of the coordinate system is at rest relative to space (4.8). $r^2 + w^2 = s^2$ expresses the relations involved and A will see s as a straight line connecting the origin with the position of B at any instant, regardless of whether B considers himself at rest or in motion.



Thus the distance s is independent of the particular coordinate system chosen. Suppose the coordinate system r,w to be rotated through an (imaginary) angle ϑ (figure 2); then the new system (denoted by r',w') is connected with the original (r,w) by the equations:

$$\left.\begin{aligned} r' &= w \sin\vartheta + r \cos\vartheta \\ w' &= w \cos\vartheta - r \sin\vartheta \end{aligned}\right\} \qquad (9.2)$$

It will be seen below that it will lead to $v = r/t$ if we let ϑ equal $\cos^{-1}(c/(c^2 - v^2)^{1/2}) = \cos^{-1} c/\nu$. ($\nu$ will hereafter stand for $\sqrt{c^2 - v^2}$.) Since $\cos\vartheta = c/\nu$, then $\sin\vartheta = \sqrt{1 - \frac{c^2}{c^2 - v^2}} = \sqrt{-\frac{v^2}{\nu^2}} = \frac{iv}{\nu}$. Therefore $\tan\vartheta = iv/c$.

Anywhere along the new w' axis, $r' = 0$ and $w \sin\vartheta = - r \cos\vartheta$ (see equation 9.2), therefore $\tan\vartheta = -r/w$. But since $\tan\vartheta$ is also iv/c, it follows that $iv/c = -r/w = ir/ct$. Therefore $v = r/t$. In other words, if B remains on the w' axis, he may think of himself as at rest ($r' = 0$), and in so doing he is employing the coordinate system r',w' in which the w' for the new position is equal to s. If, however, B thinks of himself as moving with velocity v in the direction r, then he is using the coordinate system r,w, and s is now equal to $\sqrt{r^2 + w^2}$. Consequently $r^2 + w^2 = r'^2 + w'^2$ and the equations 9.2 represent the transformation from one coordinate system to the other.

Putting the values of $\sin\vartheta$ and $\cos\vartheta$ into 9.2, we obtain

$$\left.\begin{aligned} r' &= rc/\nu + wiv/\nu \\ w' &= wc/\nu - riv/\nu \end{aligned}\right\} \qquad (9.9)$$

These equations are referred to as the Lorentz transformation. Ex-

pressed in terms of t and t' instead of w and w' they become:

$$\left.\begin{aligned} r' &= (c/\sqrt{})r - (cv/\sqrt{})t \\ t' &= (c/\sqrt{})t - (v/c\sqrt{})r \end{aligned}\right\} \qquad (10.1)$$

GENERALIZATION OF LORENTZ EQUATIONS TO ACCORD WITH D. C. MILLER'S WORK.

The Lorentz equations could have been derived by assuming the relation $r'^2 + w'^2 = r^2 + w^2$ in the first place. In order to make the later argument clearer, this process will now be carried through. It is more convenient now to drop into three dimensions and use t's instead of w's. In this form the equation just mentioned is

$$r'^2 - c^2t'^2 = r^2 - c^2t^2 \qquad (11.3)$$

In accordance with our temporary abandonment of four dimensions, we now visualize two three-dimensional coordinate systems, one moving with respect to the other. To fix our ideas, consider the unprimed system at rest and the primed system moving, with r and r' along the same line.

At the origin of the primed system, r' is permanently zero, and this origin has a velocity with respect to the unprimed system expressed by $r = vt$. These two statements may be combined into the equation

$$r' = A(r - vt) \qquad (11.6)$$

where A is a constant to be determined. Also let

$$t' = Br + Ct \qquad (11.7)$$

where B and C are likewise constants to be determined.

Substituting the values of r' and t' given by (11.6) and (11.7) in the left-hand side of (11.3) we have

$$r^2(A^2 - c^2B^2 - 1) - 2rt(A^2v + c^2BC) + t^2(A^2v^2 - c^2C^2 + c^2) = 0 \qquad (11.9)$$

Since this equation is to hold for all values of r and t, the three

parentheses must separately equal zero. The solutions of these three equations are $A = C = \pm c/v$, and $B = \mp v/cv$. It is natural to give r the same sign as r', therefore we shall take the positive sign for A and C, and this involves the negative sign for B. The result may therefore be expressed as equations (10.1).

Using the results of D. C. Miller's experiments (see 5.5) we have for the observed velocity of light, c - kv, on a system moving with an absolute velocity v in the direction in which the beam of light travels. We shall hereafter refer to the observer on this system as B". An observer at rest with respect to space will be called B, and B' will be defined presently. B will observe the velocity of light to be c. In equation form, the transmission of a light signal in B's system will be represented by r = ct, while in that of B" the equation will be r" = (c - kv)t" or r" + kvt" = ct". In the squared form these equations are $r^2 = c^2t^2$ or $r^2 - c^2t^2 = 0$, and $(r" + kvt")^2 = c^2t"^2$ or $(r" + kvt")^2 - c^2t"^2 = 0$. If we had squared the equation r" = (c - kv)t" without first transposing the term -kvt" , the modification of the Minkowski space-time continuum introduced by Miller's work would have involved the time axis. As it is, the changes occur in the direction of r and not in the direction of w and are more easily handled. Setting

$$(r" + kvt")^2 - c^2t"^2 = r^2 - c^2t^2 \qquad (12.8)$$

gives us an equation analogous to (11.3). Replacing r" by A(r - vt) and t" by Br + Ct, we obtain

$$r^2(A^2 + k^2v^2B^2 + 2kvAB - c^2B^2 - 1) - 2rt(vA^2 - kvAC + v^2kAB - k^2v^2BC + c^2BC) + t^2(A^2v^2 + k^2v^2C^2 - 2kv^2AC - c^2C^2 + c^2) = 0 \qquad (12.9)$$

As in the other case, these three parentheses must separately equal zero. Solving the three equations thus obtained and choosing the signs as before, we obtain, as may easily be verified, $A = \lambda/\nu$, $B = -k'v/\lambda\nu$, and $C = \mu^2/\lambda\nu$, where $\lambda = \sqrt{c^2 - k^2v^2}$, $\mu = \sqrt{c^2 - kv^2}$, and $k' = 1 - k = 0.95$.

Thus the equations corresponding to the Lorentz transformation become

$$\left.\begin{aligned} r'' &= \frac{\lambda}{\nu}(r - vt) \\ t'' &= \frac{\mu^2}{\lambda\nu}t - \frac{k'v}{\lambda\nu}r \end{aligned}\right\} \qquad (13.3)$$

Solved for r and t, these equations become

$$\left.\begin{aligned} r &= \frac{\mu^2}{\lambda\nu}r'' + \frac{\lambda v}{\nu}t'' \\ t &= \frac{\lambda}{\nu}t'' + \frac{k'v}{\lambda\nu}r'' \end{aligned}\right\} \qquad (13.4)$$

We shall refer to this as the new transformation. It will be noticed that when k is set equal to zero, $\lambda = \mu = c$, $k' = 1$, and these equations reduce to (10.1), the Lorentz transformation.

If, on the other hand, k be set equal to unity, then $\lambda = \mu = \nu$, and $k' = 0$. The equations then become, $r'' = r - vt$ and $t'' = t$, which are the Galilei or Newtonian "relativity" transformation equations.

A modification of this new transformation is interesting. Set

$$\left.\begin{aligned} r' &= r'' + kvt'' \\ t' &= t'' \end{aligned}\right\} \qquad (13.8)$$

These are equivalent to the equations

$$\left.\begin{aligned} r'' &= r' - kvt' \\ t'' &= t' \end{aligned}\right\} \qquad (13.9)$$

Eliminating r" and t" from equations (13.3) and (13.8) we obtain

$$r' = \frac{\mu^2}{\lambda\nu} r - \frac{k'c^2v}{\lambda\nu} t$$
$$t' = \frac{\mu^2}{\lambda\nu} t - \frac{k'v}{\lambda\nu} r \qquad (14.1)$$

These, solved for r and t are

$$r = \frac{\mu^2}{\lambda\nu} r' + \frac{k'c^2v}{\lambda\nu} t'$$
$$t = \frac{\mu^2}{\lambda\nu} t' + \frac{k'v}{\lambda\nu} r' \qquad (14.3)$$

For reasons that will presently be apparent, (13.8) and (13.9) will be called the k transformation and (14.1) and (14.3) the k' transformation. The observer corresponding with the singly primed system will be called B'.

If in (14.1) and (14.3) t and t' are replaced by w/ic and w'/ic, the k' transformation becomes

$$r' = \frac{\mu^2}{\lambda\nu} r + \frac{ck'iv}{\lambda\nu} w$$
$$w' = \frac{\mu^2}{\lambda\nu} w - \frac{ck'iv}{\lambda\nu} r \qquad (14.5)$$

(14.5) when solved for r and w become

$$r = \frac{\mu^2}{\lambda\nu} r' - \frac{ck'iv}{\lambda\nu} w'$$
$$w = \frac{\mu^2}{\lambda\nu} w' + \frac{ck'iv}{\lambda\nu} r' \qquad (14.7)$$

Representing $\mu^2/\lambda\nu$ by A and ck'iv/$\lambda\nu$ by B, (14.5) and (14.7) become

$$r' = Ar + Bw$$
$$w' = Aw - Br \qquad (14.8)$$

$$r = Ar' - Bw'$$
$$w = Aw' + Br' \qquad (14.9)$$

The relations between the unprimed, the primed and the doubly primed systems may be visualized in the following way.

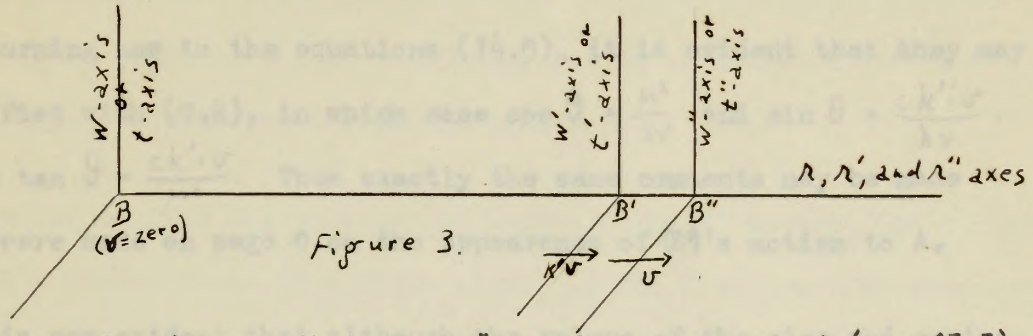
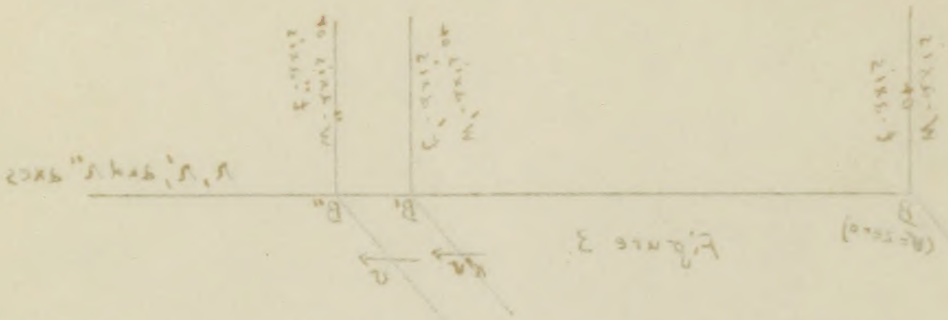


Figure 3.

We have already seen that when r" remains zero, r = vt (see 13.3), therefore B" moves to the right in figure 3 relative to B with velocity v. From (13.8) it is evident that when r' is zero, r" = -kvt", therefore B' moves to the left with an approximate velocity -kv relative to B", justifying the term "k transformation" (14.4). Thus B' moves to the right relative to B with an approximate velocity (1 - k)v or k'v, furnishing the reason for the term "k' transformation" (14.4). The word approximate is used because the -kv is the value of r"/t" rather than of r/t. The exact velocity of B' may be obtained from (14.1): when r' = 0, $r/t = \frac{c^2}{\mu^2}k'v$. These values have been stated merely to help visualize the relations between the three systems; the precise values to be used depend of course on the purpose at hand as will be illustrated presently.

If k be set equal to zero, it will be seen that the k' transformation (as well as the "new transformation") reduces to the Lorentz transformation. That is to say, when k = 0, B' and B" become identical.

If k be set equal to unity (in which case k' is zero), the k' transformation reduces to the identical transformation. In this case r' = r and t' = t.



It will be seen presently that the outstanding property of the B' system is that the velocity of light relative to it is always c.

Returning now to the equations (14.5), it is evident that they may be identified with (9.2), in which case $\cos\vartheta = \frac{\mu^2}{\lambda\nu}$ and $\sin\vartheta = \frac{ck'iv}{\lambda\nu}$. Therefore $\tan\vartheta = \frac{ck'iv}{\mu^2}$. Thus exactly the same comments may be made again as were made on page 9 on the appearance of (B')'s motion to A.

It is now evident that although the values of the sine and cosine are slightly different from formerly, the whole structure which has been erected upon the Lorentz transformation may be transferred to the k' transformation, and that the consequence of assuming a null result of the Michelson-Morley experiment was merely to identify B" with B'.

It must be emphasized B' is a fictitious observer, resembling the astronomer's fictitious sun which behaves more simply than the actual sun.

The necessary modification of Minkowski's four-dimensional space-time continuum consistent with the results of D. C. Miller's repetition of the Michelson-Morley experiment is therefore two-fold: it first consists in replacing the actual observer B" by a fictitious observer B' by means of the k transformation equations (13.8 and 13.9), and then in slightly changing the value of the angle through which Minkowski rotated his four-dimensional axes.

It will be seen presently that the outstanding property of the S'
system is that the velocity of light relative to it is always c.

Returning now to the equations (14.3), it is evident that they may
be identified with (6.2), in which case $\cos\theta = \frac{v}{\lambda\nu}$ and $\sin\theta = \frac{ck'i\nu}{\lambda\nu}$.
Therefore $\tan\vartheta = \frac{ck'i\nu}{v}$. Thus exactly the same comments may be made
again as were made on page 6 on the appearance of ϑ''s motion to A.

It is now evident that although the values of the sine and cosine
are slightly different from formerly, the whole structure which has been
erected upon the Lorentz transformation may be transferred to the k'
transformation, and that the consequence of assuming a null result of
the Michelson-Morley experiment was merely to identify S' with S_1.

It must be emphasized S' is a fictitious observer, resembling the
astronomer's fictitious sun which behaves more simply than the actual
sun.

The necessary modification of Minkowski's four-dimensional space-
time continuum consistent with the results of D. C. Miller's repetition
of the Michelson-Morley experiment is therefore two-fold: it first con-
sists in replacing the actual observer S" by a fictitious observer S'
by means of the k transformation equations (14.3 and 14.5), and then in
slightly changing the value of the angle through which Minkowski rotated
his four-dimensional axes.

VELOCITY OF LIGHT IN THE THREE SYSTEMS

This section will serve two purposes. It will not only determine the velocity of light in the three systems but it will also serve to check the three sets of transformation equations.

In order to find the velocity of light in the B' system, let r = ct refer to the transmission of a light signal in the B system (the absolute system). In the k' transformation (14.1) replace r by ct and obtain

$$\begin{aligned} r' &= \frac{ct}{\lambda\nu}(\mu^2 - k'cv) \\ t' &= \frac{t}{\lambda\nu}(\mu^2 - k'cv) \end{aligned} \qquad (17.4)$$

The ratio of r' to t' will give the velocity in the B' system. r'/t' = c. At (16.0) reference has already been made to this fact.

Now reverse the procedure and assume r' = ct' for the B' system. Substituting this value of r' in (14.3) we have

$$\begin{aligned} r &= \frac{ct'}{\lambda\nu}(\mu^2 + k'cv) \\ t &= \frac{t'}{\lambda\nu}(\mu^2 + k'cv) \end{aligned} \qquad (17.6)$$

The ratio of r to t is c as was to be expected.

Passing from the B' system to the B" system, we start with r' = ct' and substitute in (13.9):

$$\begin{aligned} r'' &= ct' - kvt' \\ t'' &= t' \end{aligned} \qquad (17.9)$$

Therefore r"/t" = c - kv in accordance with our original hypothesis.

Passing in the opposite direction from the B" system to the B'

system, we start with r" = (c - kv)t" and substitute in (13.8), obtaining

$$\begin{aligned} r' &= ct'' - kvt'' + kvt'' \\ t' &= t'' \end{aligned} \qquad (18.1)$$

and r'/t' = c again.

The remaining check consists in passing from the B system to the B" system and back again. Setting r = ct and substituting in (13.3), we have

$$\begin{aligned} r'' &= \frac{\lambda^2 t}{\lambda\nu}(c - v) \\ t'' &= \frac{t}{\lambda\nu}(\mu^2 - k'cv) \end{aligned} \qquad (18.3)$$

$$\frac{r''}{t''} = \frac{(c^2 - k^2v^2)(c - v)}{c^2 - kv^2 - cv + kcv} = \frac{(c - kv)(c + kv)(c - v)}{(c + kv)(c - v)} = c - kv \text{ as expected.}$$

Finally set r" = (c - kv)t" and substitute in the equations (13.4).

$$\begin{aligned} r &= \frac{t''}{\lambda\nu}(\mu^2 c - \mu^2 kv + \lambda^2 v) \\ t &= \frac{t''}{\lambda\nu}(\lambda^2 + ck'v - kk'v^2) \end{aligned} \qquad (18.6)$$

$$\frac{r}{t} = \frac{c^3 - ckv^2 - c^2kv + k^2v^3 + c^2v - k^2v^3}{c^2 - k^2v^2 + cv - ckv - kv^2 + k^2v^2} = c, \text{ which completes the checks.}$$

In other words, the velocity of light in the unprimed system is c by hypothesis, corresponding to the statement that the "absolute velocity" of light is c. Also by hypothesis the velocity of light relative to the doubly primed system is c - kv. By means of the transformation equations we have established the fact that the velocity of light relative to the singly primed system is c, and as a check we have shown that the other two velocities follow from the transformation equations.

LIMITING VELOCITY

The upper limit of signal velocities is still c in the case of the new equations. Consider a rod the length of which is unity in the B" coordinate system, and let the ends be located permanently at the points r" = 0 and r" = 1. Find the position of the ends in the B system at the time t = 0. This will give us the length of the rod at that time in the absolute system. Substituting in the first of the two equations (13.3), we find that when r" = 0, r = 0, and when r" = 1, $r = \nu/\lambda = \sqrt{\frac{c^2 - v^2}{c^2 - k^2v^2}}$ The radical reduces to zero when v becomes c and c thus becomes the limiting velocity.

If, however, we express the length of a unit rod in the B system in the B" coordinates, we obtain for the new length (using 13.4) $\lambda\nu/\mu^2$. But ν is still in the numerator so that the limiting velocity is still c.

It may be shown that a unit length in the B system will transform into $\frac{\lambda\nu}{\mu^2}$ in the B' system, and this time the transformation is reciprocal; that is, unit length in the B' system becomes $\frac{\lambda\nu}{\mu^2}$ in the B system. And the limiting velocity is still c.

A transformation from B' to B" or the reverse involves no change in length.

A seconds pendulum permanently located at r" = 0 emitting a tick at t" = 0 and another at t" = 1 will emit these two ticks according to the unprimed system at t = 0 and $t = \frac{\lambda}{\nu} = \sqrt{\frac{c^2 - k^2v^2}{c^2 - v^2}}$ by (13.4). There would be an infinite time between ticks when v = c.

Similarly a seconds clock permanently located at r = 0, emitting

ticks at t = 0 and t = 1, will emit these two ticks at t" = 0 and t" = $\frac{\mu^2}{\lambda v}$ by (13.3). This interval also becomes infinite when v = c.

There is a reciprocal transformation in the time unit between the B and the B' systems, unit time in either case becoming $\frac{\mu^2}{\lambda v}$.

And as in the case of the distance unit, a transformation from B' to B" or the reverse involves no change in the time unit.

DETERMINATION OF V

Thus far we have assumed that v is measured in terms of absolute (unprimed) coordinates. What difference will it make if we express v in terms of the doubly primed system?

Let the coordinates of an object at rest relative to the B system (for convenience placed at the origin so that r is permanently zero) be transformed to those of the B" system. Then, by (13.3), $r'' = -\frac{\lambda}{\nu}vt$ and $t'' = \frac{\mu^2 t}{\lambda\nu}$ so that $r''/t'' = -\frac{\lambda^2}{\mu^2}v$; in other words, B" has a velocity relative to B of v, but B has a velocity relative to B" of $-\frac{\lambda^2}{\mu^2}v$!

If the problem be tried the other way around, so that r" is permanently equated to zero, using (13.4), we have $r = \frac{\lambda v}{\nu}t''$ and $t = \frac{\lambda}{\nu}t''$ so that r/t reduces to v as we should expect it to.

These results emphasize the "quasi"-ness of the relativity with which we are now dealing.

Let us try the same experiment with the B and B' systems, first assuming that r is permanently zero. From (14.1) we have $r' = -\frac{k'c^2vt}{\lambda\nu}$ and $t' = \frac{\mu^2 t}{\lambda\nu}$ so that $r'/t' = -k'\frac{c^2}{\mu^2}v$. The other way about, letting r' = 0 and using (14.3) gives $r/t = k'\frac{c^2}{\mu^2}v$ as we should expect, in keeping with the relativistic properties of the B' system.

Returning to the expression derived above at (21.3) namely $\frac{\lambda''}{t''} = -\frac{\lambda^2 v}{\mu^2}$, if we represent the velocity r"/t" as observed by B" by -v", we have $\lambda^2 v = \mu^2 v''$ or $c^2v - k^2v^3 = c^2v'' - kv^2v''$, or $k^2v^3 - kv''v^2 - c^2v + c^2v'' = 0$, a cubic in v.

Thus, if B" could observe space streaming by with a velocity -v",

he could, by solving this cubic, compute v, the velocity with which his system is moving relative to space. This, however, is not an experimental possibility.

The experimental procedure is as follows. In the B" system, the velocity of light is measured and found to be c_1 (= c - kv) in the direction of motion of the system, and c_2 (= c + kv) in the opposite direction. Therefore $(c_1 + c_2)/2 = c$ and $(c_2 - c_1)/2k = v$. Thus, knowing k, v may be computed. Afterward, as a means of entertainment, v" ($= \frac{\lambda^2}{\mu^2} v$) may be computed, although it has no practical value, since all our equations have been expressed in terms of v.

TRANSFORMATION FROM ONE MOVING SYSTEM TO ANOTHER

It should of course be possible to transform directly from one doubly primed system to another without using either the unprimed (absolute) or the singly primed systems as intermediaries.

Let us derive the transformation equations necessary to pass from one singly primed system to another. From (14.9) we have

$$\begin{aligned} r &= A_1 r_1' - B_1 w_1' \\ w &= A_1 w_1' + B_1 r_1' \end{aligned} \qquad (23.3)$$

which transforms from one of the singly primed systems to the absolute system; also

$$\begin{aligned} r &= A_2 r_2' - B_2 w_2' \\ w &= A_2 w_2' + B_2 r_2' \end{aligned} \qquad (23.5)$$

which transforms from the other singly primed system. It may easily be shown that $A_1^2 + B_1^2 = 1 = A_2^2 + B_2^2$. If we let $C_{12} = A_1 A_2 + B_1 B_2$ and $D_{12} = A_2 B_1 - A_1 B_2$, we may eliminate r and w from equations (23.3) and (23.5) and obtain

$$\begin{aligned} r_1' &= C_{12} r_2' + D_{12} w_2' \\ w_1' &= C_{12} w_2' - D_{12} r_2' \end{aligned} \qquad (23.8)$$

and, solving for r_2' and w_2'

$$\begin{aligned} r_2' &= C_{12} r_1' - D_{12} w_1' \\ w_2' &= C_{12} w_1' + D_{12} r_1' \end{aligned} \qquad (23.9)$$

The two coefficients C_{12} and D_{12} may be shown to have the values

$$C_{12} = \frac{\nu_{12}^2 \lambda_{12}^2 - kc^2 v_{12}^2}{\lambda_1 \lambda_2 \nu_1 \nu_2} \quad \text{and} \quad D_{12} = \frac{cik' \mu_{12}'^2 v_{12}}{\lambda_1 \lambda_2 \nu_1 \nu_2} \quad \text{where } v_{12} = v_1 - v_2$$

$$\nu_{12} = \sqrt{c^2 - v_1 v_2} \qquad \lambda_{12} = \sqrt{c^2 - k^2 v_1 v_2} \quad \text{and} \quad \mu_{12}' = \sqrt{c^2 + k v_1 v_2}$$

It is readily seen that these equations reduce to (14.8) or (14.9) if either v_1 or v_2 is zero. If, for example, $v_2 = 0$, then $v_{12} = v_1$, and $\lambda_2 = \mu_2 = \nu_2 = \nu_{12} = \lambda_{12} = \mu'_{12} = c$. It will follow that $C_{12} = A_1 = \frac{\mu_1^2}{\lambda_1 \nu_1}$, and $D_{12} = B_1 = \frac{cik'v_1}{\lambda_1 \nu_1}$.

Similarly it is possible to write for two doubly primed systems:

$$\begin{aligned} r_2'' &= a_{21} r_1'' + b_{21} t_1'' \\ t_2'' &= c_{21} r_1'' + d_{21} t_1'' \end{aligned} \qquad (24.4)$$

and the corresponding equations

$$\begin{aligned} r_1'' &= a_{12} r_2'' + b_{12} t_2'' \\ t_1'' &= c_{12} r_2'' + d_{12} t_2'' \end{aligned} \qquad (24.5)$$

where $a_{21} = d_{12} = \frac{\rho_{12}^2 \lambda_2^2}{\lambda_1 \lambda_2 \nu_1 \nu_2}$, $b_{21} = -b_{12} = -\frac{v_{21} \lambda_1^2 \lambda_2^2}{\lambda_1 \lambda_2 \nu_1 \nu_2}$, $c_{21} = -c_{12} = -\frac{v_{21} \mu_{21}'^2 k'}{\lambda_1 \lambda_2 \nu_1 \nu_2}$

$a_{12} = d_{21} = \frac{\rho_{21}^2 \lambda_1^2}{\lambda_1 \lambda_2 \nu_1 \nu_2}$, and $\rho_{12} = \sqrt{\mu_1^2 - k' v_1 v_2}$, hence $\rho_{21} = \sqrt{\mu_2^2 - k' v_2 v_1}$

The velocity of the observer B_2'' as seen by B_1'' may be found by setting r_2'' permanently equal to zero in (24.5) which gives $r_1'' = b_{12} t_2''$ and $t_1'' = d_{12} t_2''$, whence $r_1''/t_1'' = b_{12}/d_{12} = \frac{v_{21} \lambda_1^2}{\rho_{12}^2}$ (24.8)

In similar manner from (24.4) the velocity of B_1'' as seen by B_2'' may be derived. It comes out $r_2''/t_2'' = v_{12} \frac{\lambda_2^2}{\rho_{21}^2}$ (24.9)

ILLUSTRATION

As an illustration of the use of these equations, consider the following situation:

Observer B_1'' sends a light signal to B_2'' which is reflected back from the latter's system so as to arrive again at B_1''. Find the relationships involved, assuming the velocity of B_1'' to be v_1 and that of B_2'' to be v_2 relative to space.

Draw three diagrams using successively the three assumptions, (1) Both B_1'' and B_2'' relative to a stationary observer, A, are moving with velocities v_1 and v_2; (2) B_1'' thinks of himself as at rest and of B_2'' as moving with velocity $v_{21}\frac{\lambda_1^2}{\rho_{12}^2}$ (see 24.8); (3) B_2'' thinks of himself as at rest and of B_1'' as moving with velocity $v_{12}\frac{\lambda_2^2}{\rho_{21}^2}$ (see 24.9).

From point of view of stationary observer, A.

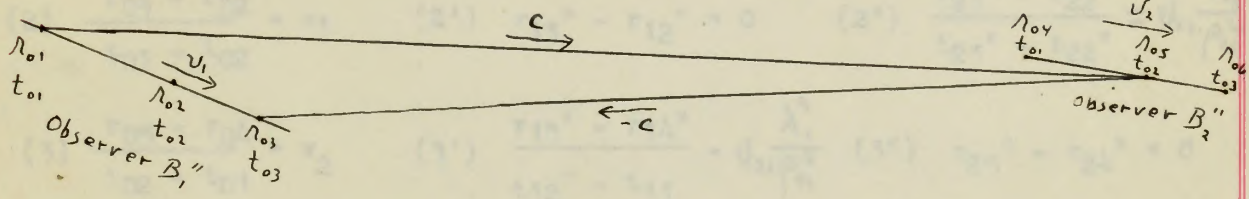


Figure 4.

From point of view of B_1''

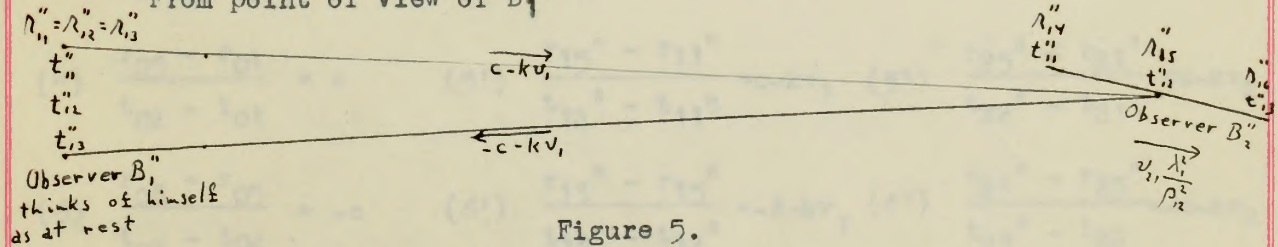


Figure 5.

From point of view of B_2''

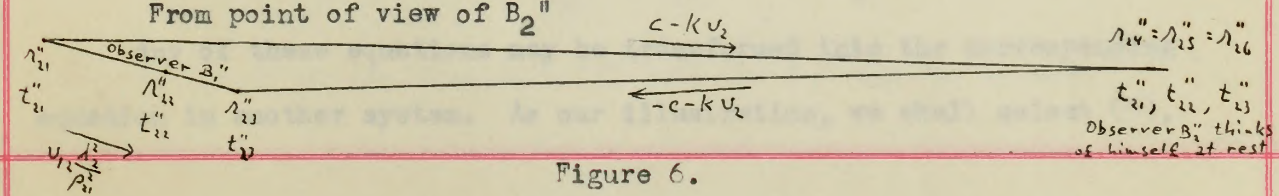


Figure 6.

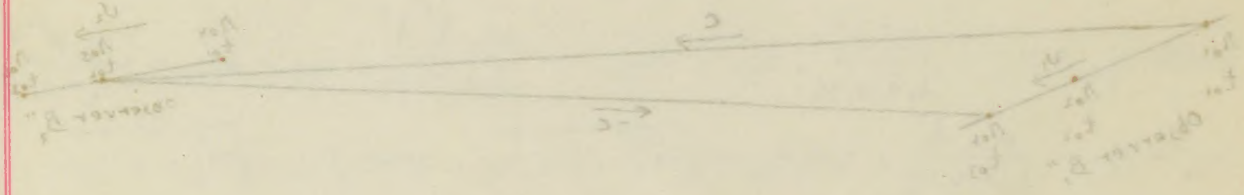


Figure 4.

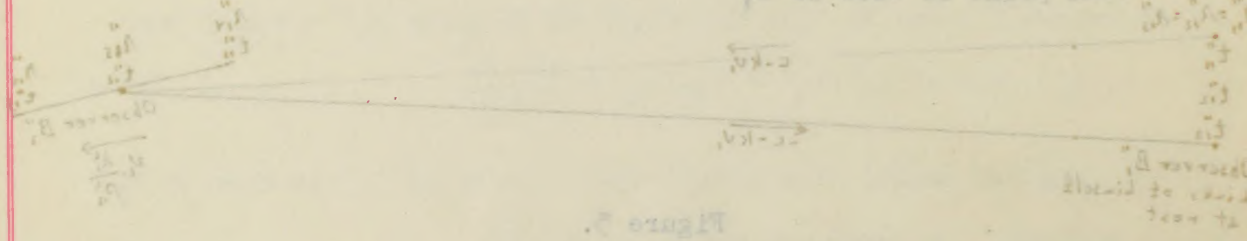


Figure 5.

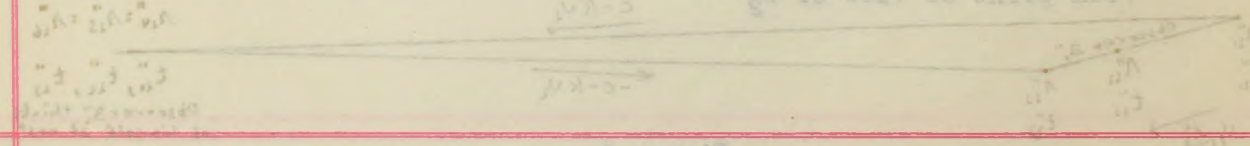


Figure 6.

Each observer sets up his equations on the basis that his own system is at rest. In this problem, $B_1"$ is the only physically possible observer; we must use our imaginations in regard to the other two cases. Although $B_1"$ considers himself at rest, he must compute the velocity of light as $\pm c - kv_1$. Intuitive ideas of simultaneity are employed in setting up the equations. For example, $B_1"$ considers his position $r_{12}"$ and the position $r_{15}"$ belonging to $B_2"$ as occupied at simultaneous times.

Our illustration will consist of setting up the equations from all three points of view and showing that an equation from one set may be transformed into one in another system by using the appropriate equations.

From point of view of A	From point of view of $B_1"$	From point of view of $B_2"$
(1) $\frac{r_{02} - r_{01}}{t_{02} - t_{01}} = v_1$	(1') $r_{12}" - r_{11}" = 0$	(1") $\frac{r_{22}" - r_{21}"}{t_{22}" - t_{21}"} = v_{12}\frac{\lambda_2^2}{\rho_{21}^2}$
(2) $\frac{r_{03} - r_{02}}{t_{03} - t_{02}} = v_1$	(2') $r_{13}" - r_{12}" = 0$	(2") $\frac{r_{23}" - r_{22}"}{t_{23}" - t_{22}"} = v_{12}\frac{\lambda_2^2}{\rho_{21}^2}$
(3) $\frac{r_{05} - r_{04}}{t_{02} - t_{01}} = v_2$	(3') $\frac{r_{15}" - r_{14}"}{t_{12}" - t_{11}} = v_{21}\frac{\lambda_1^2}{\rho_{12}^2}$	(3") $r_{25}" - r_{24}" = 0$
(4) $\frac{r_{06} - r_{05}}{t_{03} - t_{02}} = v_2$	(4') $\frac{r_{16}" - r_{15}"}{t_{13}" - t_{12}"} = v_{21}\frac{\lambda_1^2}{\rho_{12}^2}$	(4") $r_{26}" - r_{25}" = 0$
(5) $\frac{r_{05} - r_{01}}{t_{02} - t_{01}} = c$	(5') $\frac{r_{15}" - r_{11}"}{t_{12}" - t_{11}"} = c - kv_1$	(5") $\frac{r_{25}" - r_{21}"}{t_{22}" - t_{21}} = c - kv_2$
(6) $\frac{r_{03} - r_{05}}{t_{03} - t_{02}} = -c$	(6') $\frac{r_{13}" - r_{15}"}{t_{13}" - t_{12}"} = -c - kv_1$	(6") $\frac{r_{23}" - r_{25}"}{t_{23}" - t_{22}} = -c - kv_2$

Any of these equations may be transformed into the corresponding equation in another system. As our illustration, we shall select (5),

transform it not only into (5') and (5") but into the corresponding B' systems as well. From (13.4) we have

$$\begin{aligned} r_{05} &= \frac{\mu_1^2}{\lambda_1\nu_1} r_{15}'' + \frac{\lambda_1 v_1}{\nu_1} t_{12}'' \\ r_{01} &= \frac{\mu_1^2}{\lambda_1\nu_1} r_{11}'' + \frac{\lambda_1 v_1}{\nu_1} t_{11}'' \\ t_{02} &= \frac{\lambda_1}{\nu_1} t_{12}'' + \frac{k'v_1}{\lambda_1\nu_1} r_{15}'' \\ t_{01} &= \frac{\lambda_1}{\nu_1} t_{11}'' + \frac{k'v_1}{\lambda_1\nu_1} r_{11}'' \end{aligned} \qquad (27.2)$$

Clearing (5) of fractions and substituting these values, we have

$$\frac{\mu_1^2}{\lambda_1\nu_1}\lambda_{15}'' + \frac{\lambda_1 v_1}{\nu_1} t_{12}'' - \frac{\mu_1^2}{\lambda_1\nu_1}\lambda_{11}'' - \frac{\lambda_1 v_1}{\nu_1} t_{11}'' = \frac{c\lambda_1}{\nu_1} t_{12}'' + \frac{ck'v_1}{\lambda_1\nu_1}\lambda_{15}'' - \frac{c\lambda_1}{\nu_1} t_{11}'' - \frac{ck'v_1}{\lambda_1\nu_1}\lambda_{11}''$$

Multiply through by $\lambda_1\nu_1$ and expand:

$$(c^2 - kv_1^2)\lambda_{15}'' + (c^2 - k^2v_1^2)v_1 t_{12}'' - (c^2 - kv_1^2)\lambda_{11}'' - (c^2 - k^2v_1^2)v_1 t_{11}''$$

$$= (c^2 - k^2v_1^2)c t_{12}'' + cv_1\lambda_{15}'' - ckv_1\lambda_{15}'' - (c^2 - k^2v_1^2)c t_{11}'' - cv_1\lambda_{11}'' + ckv_1\lambda_{11}''$$

Group the terms:

$$\lambda_{15}''(c^2 - kv_1^2 - cv_1 + ckv_1) - \lambda_{11}''(c^2 - kv_1^2 - cv_1 + ckv_1) =$$

$$= t_{12}(c^3 - ck^2v_1^2 - c^2v_1 + k^2v_1^3) + t_{11}''(c^2v_1 - k^2v_1^3 - c^3 + ck^2v_1^2)$$

Factor and cancel:

$$\lambda_{15}''(c+kv_1)(c-v_1) - \lambda_{11}''(c+kv_1)(c-v_1) = t_{12}(c+kv_1)(c-kv_1)(c-v_1) - t_{11}''(c+kv_1)(c-kv_1)(c-v_1)$$

The result is therefore

$$r_{15}'' - r_{11}'' = (t_{12}'' - t_{11}'')(c - kv_1)$$

And similarly,

$$r_{25}'' - r_{21}'' = (t_{22}'' - t_{21}'')(c - kv_2)$$

By (14.3) we may transform (5) into the B_1' and B_2' systems:

TO B_1'

$$r_{05} = \frac{\mu_1^2}{\lambda_1 \nu_1} r'_{15} + \frac{c^2 k' v_1}{\lambda_1 \nu_1} t'_{12}$$

$$r_{01} = \frac{\mu_1^2}{\lambda_1 \nu_1} r'_{11} + \frac{c^2 k' v_1}{\lambda_1 \nu_1} t'_{11}$$

$$t_{02} = \frac{\mu_1^2}{\lambda_1 \nu_1} t'_{12} + \frac{k' v_1}{\lambda_1 \nu_1} r'_{15}$$

$$t_{01} = \frac{\mu_1^2}{\lambda_1 \nu_1} t'_{11} + \frac{k' v_1}{\lambda_1 \nu_1} r'_{11}$$

TO B_2'

$$r_{05} = \frac{\mu_2^2}{\lambda_2 \nu_2} r'_{25} + \frac{c^2 k' v_2}{\lambda_2 \nu_2} t'_{22}$$

$$r_{01} = \frac{\mu_2^2}{\lambda_2 \nu_2} r'_{21} + \frac{c^2 k' v_2}{\lambda_2 \nu_2} t'_{21}$$

$$t_{02} = \frac{\mu_2^2}{\lambda_2 \nu_2} t'_{22} + \frac{k' v_2}{\lambda_2 \nu_2} r'_{25}$$

$$t_{01} = \frac{\mu_2^2}{\lambda_2 \nu_2} t'_{21} + \frac{k' v_2}{\lambda_2 \nu_2} r'_{21}$$

Clearing (5) of fractions and substituting these values, we have for the

B_1' system:

$$\mu_1^2 r'_{15} + c^2 k' v_1 t'_{12} - \mu_1^2 r'_{11} - c^2 k' v_1 t'_{11} = c\mu_1^2 t'_{12} + c k' v_1 r'_{15} - c\mu_1^2 t'_{11} - c k' v_1 r'_{11}$$

$$r'_{15}(\mu_1^2 - c k' v_1) - r'_{11}(\mu_1^2 - c k' v_1) = c\, t'_{12}(\mu_1^2 - c k' v_1) - c\, t'_{11}(\mu_1^2 - c k' v_1)$$

$$\frac{r_{15}' - r_{11}'}{t_{12}' - t_{11}'} = c$$

And similarly for the B_2' system:

$$\frac{r_{25}' - r_{21}'}{t_{22}' - t_{21}'} = c$$

APPLICATION TO FIZEAU'S EXPERIMENT.

Fizeau caused water to have a motion relative to the containing tubes, and also gave light a motion relative to the water. By a differential method he experimentally determined the velocity of the light relative to the tubes. Our problem is to compute this velocity and compare our result with the experimental results of Fizeau and his successors.

Let v be the velocity of the water relative to the containing tube, c/n the velocity of light relative to the water, and u the velocity of light relative to the tubes.

As a first approximation, let us use the Lorentz transformation, $r' = \frac{c}{v} r - \frac{cv}{v} t$ and $t' = \frac{ct}{v} - \frac{vr}{cv}$. The r', t' system is at rest with reference to the water, and $r' = (c/n)t'$. Substituting the first two equations into the third, we obtain

$$\frac{c}{v} r - \frac{cv}{v} t = \frac{c^2 t}{nv} - \frac{vr}{nv}$$

Setting $r = ut$ and solving for u, the result is

$$u = (cnv + c^2)/(cn + v) \qquad (29.7)$$

Next use the new transformation. Our three equations are (13.3) and $r'' = t''(c - kv)/n$. Making the substitution we obtain

$$\frac{\lambda r}{v} - \frac{\lambda}{v} vt = \left(\frac{c - kv}{\eta}\right)\left(\frac{\mu^2 t}{\lambda v} - \frac{k'v}{\lambda v} r\right)$$

Setting $r = ut$ and solving for u, the result is

$$u = \frac{c^2\mu^2 - k\mu^2 cv + nc\lambda^2 v}{nc\lambda^2 - kk'cv^2 + c^2 k'v} \qquad (29.9)$$

Fresnel's equation was $u = c/n + v - v/n^2$.

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$$u = (cnv + c^2)/(cn + v) \qquad (27.7)$$

Next use the new transformation. Our three equations are (27.3) and $r' = t'(c - kv)/k$. Making the substitution we obtain

$$\frac{kr}{v} - \frac{k}{v}vt = \left(\frac{c - kv}{n}\right)\left(\frac{kt}{v} - \frac{kvr}{cv}\right)$$

Setting $r = ut$ and solving for u, the result is

$$u = \frac{c^2k^2 - k^2v^2 + nc\,k^2v}{nck^2 - nk^2v + c^2k^2}$$

(27.8)

Compare equation (27.7) with $u = c/n + v - v/n^2$.

The expansion of (29.7) by Maclaurin's theorem in powers of v gives

$$u = c/n + v(1 - 1/n^2) - (v^2/cn)(1 - 1/n^2) \text{ etc.}$$

It may therefore be regarded as identical with Fresnel's equation as far as the measurable terms are concerned, since the v^2 terms and those following them are negligible.

The expansion of (29.9) by Maclaurin's theorem is

$$u = c/n + v(1 - 1/n^2 - k(n - 1)/n^2)$$

To both of these should be added a term due to the Doppler effect, so that the two formulas become

$$u = c/n + v\left(1 - \frac{1}{n^2} - \frac{ldn}{ndl}\right) \quad \text{Fresnel, Lorentz, Einstein}$$

$$u = c/n + v\left(1 - \frac{1}{n^2} - \frac{ldn}{ndl} - \frac{k(n - 1)}{n^2}\right) \quad \text{New formula}$$

For sodium light ($l = 5893$), let $n = 1.332$, $n^2 = 1.776$, $1/n^2 = 0.563$, $-\frac{ldn}{ndl} = +0.014$, and $k(n - 1)/n^2 = 0.05(0.332)/1.776 = 0.009$.

It is interesting to review the history of this formula. Fizeau merely attempted to distinguish between a value of the parenthesis equal to 1.000 and 1.000 - 0.563 or 0.437. His experimental value was 0.460 which he considered a confirmation of Fresnel's formula.

Michelson and Morley repeated the experiment in 1886 and found the parenthesis to be 0.437 for sodium light.

In 1895 Lorentz demonstrated the existence of the Doppler term which increased the value of the parenthesis to 0.451 for sodium light.

In 1915 Zeeman again repeated the experiment and found a value cor-

responding to sodium light of 0.448. He was not quite satisfied with this value, accordingly he repeated the experiment with an improved method of determining the velocity of the water and brought the value up to 0.450, an excellent check with the theoretical 0.451.

Zeeman's experimental evidence however points to a value somewhat less than the formula as Lorentz left it. D. C. Miller's value of k makes the new addition to the parenthesis -0.009 which would bring the value back down to 0.442. The average of Zeeman's and Michelson's and Morley's result is 0.444. It is of course perfectly possible that Miller's value of k is too great, but the evidence certainly points to a value of k somewhat greater than zero.

It would be interesting to repeat the Fizeau experiment with some liquid such as carbon disulphide which has a larger index of refraction. The Miller effect in carbon disulphide would be 25% greater than in water and could perhaps be definitely confirmed.

APPLICATION TO MASS OF ELECTRON AT HIGH VELOCITIES.

When the new transformation is applied to the mass of the electron, the transverse mass comes out $m_t = m\frac{\lambda}{\nu}$ instead of $m_t = m\frac{c}{v}$ as derived from the Lorentz transformation. Thus the change of transverse mass with velocity is in accordance with the equation

$$\frac{dm_t}{dv} = \frac{mc^2v(1 - k^2)}{\lambda v^3} = \frac{mcv}{v^3}(1 - k^2)\frac{c}{\lambda}$$

instead of

$$\frac{dm_t}{dv} = \frac{mcv}{v^3}, \text{ the Lorentz value.}$$

Thus, by either formula, the limiting value is reached when $v = c$ and $\nu = 0$. When $v = 0$ the ratio $(1 - k^2)\frac{c}{\lambda}$ is $1 - k^2 = 0.998$, and when $v = c$, the ratio rises to $\sqrt{1 - k^2} = 0.999$ and has intermediate values between. In other words this ratio is a pretty good constant and there is therefore no hope of deciding between the two equations experimentally.

$m\frac{\lambda}{\nu}$ is equal approximately to $m + \frac{1 - k^2}{2} m\frac{v^2}{c^2}$ or $m + 0.499mv^2/c^2$. Thus the energy of the electron, m_tc^2 comes out equal to $mc^2 + 0.499mv^2$, approximately the same as the Lorentz value, $mc^2 + 0.500mv^2$.

SUMMARY

Our task has been a peculiar one. We were presented on the one hand with the magnificent structure of relativity, based on the hypothesis that whether space has a structure or not, it is forever impossible to detect motion with respect to it. As Einstein says, there is something so satisfying in the conceptions involved, that we should be tempted to cling to them, even without experimental evidence.

But, on the other hand, there is the formidable mass of evidence arrayed by Miller to prove that motion relative to space can be detected.

Can we accept this experimental evidence as valid and still retain the relativity structure? In other words, is the structure valid without the relativity?

The loop-hole of escape lies in the fact that the absolute motion detected is revealed, not in toto, but diluted to 5%.

This allows us to build a mathematical reference system nearly identical with the actual physical coordinate system naturally used, related to it by simple transformation equations; a system which has all the characteristics claimed by Minkowski's space-time continuum. If Miller had discovered that absolute motion could be detected in full, this mathematical reference system would have been identical with space itself, and the relativity structure would have been forced to collapse.

The ~~constants~~ parameters, however, in our new system are slightly different from Minkowski's, and this gives us our hope of experimentally settling the question by methods other than the well-known Michelson-Morley

experiment. In fact, Michelson and Morley themselves by their repetition of the Fizeau experiment have furnished more evidence in favor of Miller's results than any other experiment ever performed. It is also interesting that Zeeman had a struggle to get the result of his Fizeau experiment nearly up to the relativity requirements. It is sometimes remarkable how a theory seems to influence experimental results!

An explanation of the Miller effect lies outside of the scope of this paper. It has already been suggested that a revival of the ether entrainment hypothesis would serve the purpose. We have, however, accomplished our purpose in reconciling these two "irreconcilables" and in showing that at present at least, there seems to be no evidence against D. C. Miller's results.

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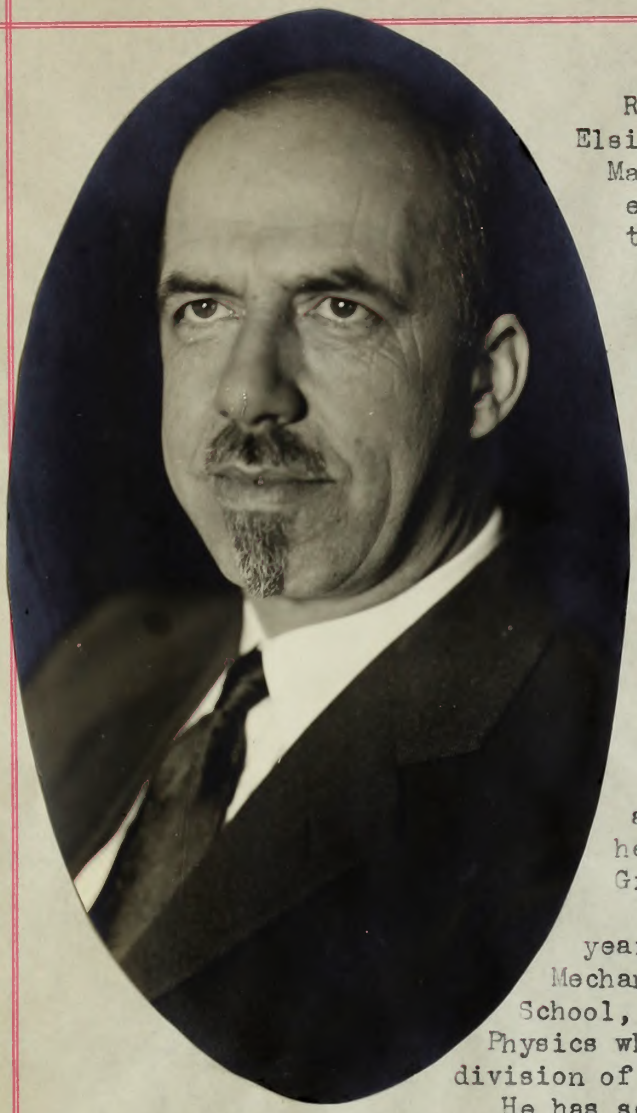

BIOGRAPHICAL

Royal Merrill Frye, son of Frank B. and Elsie W. Frye was born at Milford, N. H. May 27, 1890; received his elementary education in the public schools of Waltham, Mass., graduating from Waltham High School as valedictorian and editor-in-chief of the school paper.

He received his A. B. from B. U. in 1911 with Phi Beta Kappa honors, and his A. M. in 1912, majoring in Physics. The year following he studied at Harvard as Jacob Sleeper Fellow. In 1913-14, and 1914-15, he was acting head of the Physics and Chemistry Departments of B. U. C. L. A. in succession during the sabbatical leaves of absence of Professor Kent and the late Professor Newell.

In 1915 he was appointed Instructor in Physics at Massachusetts Institute of Technology, and in 1921 was advanced to the faculty. He continued at M. I. T. until three years ago when he registered for Ph. D. work in the Graduate School of Boston University.

In this interval he has served two years as Special Lecturer in Quantum Mechanics in the Boston University Graduate School, and has written a text in Practical Physics which is now being used in the evening division of Northeastern University.

He has served as Alderman in the Waltham city government and is an official member of the First Methodist Church of Waltham, where he also served for many years as Church School Superintendent.

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